

Generalized Dirichlet series of n variables associated with automatic sequences

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Résumé

This article consists to give a necessary and sufficient condition of the meromorphic continuity of Dirichlet series defined as $\sum_{\underline{x} \in \mathbf{N}^n} \frac{a_{\underline{x}}}{P(\underline{x})^s}$, Where $a_{\underline{x}}$ is a q -automatic sequence of n parameters and $P : \mathbf{C}^n \rightarrow \mathbf{C}$ a polynomial, such that P does not have zeros on \mathbf{Q}_+^n . And some specific cases of $n = 1$ will also be studied in this article as examples to show the possibility to have an holomorphic continuity on the whole complex plane. Some equivalences between infinite products are also built as consequences of these results.

Introduction

An automatic sequence can be defined intuitively as a sequence of multi-index with values in a finite set, such that for a given integer $d \geq 2$, each term of such sequence can be computed by using the base d expansion of the its index. The phenomena of automates can be explained by using the language of sub-sequences, which will be given as definition in the next section, as well as the language of state machine. [7] gives an example to prove the equivalence between the two definitions in the 2-index case, the statement used in the above article can be extended to an n -index case for an arbitrary entire number n . Such sequences show many interesting properties in a large span of researching fields, such as number theory, harmonic analysis, even in physics and theoretical computer science [1].

The Dirichlet sequences of the form $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ have been studied in [3], and our job is a natural generalization of the results in above article by using the same method of calculations. Remarking that the constant sequences are a kind of particular automatic sequences, the Dirichlet sequences in the form $\sum_{(n_1, n_2, \dots, n_I) \in \mathbf{N}_+^I} \frac{n_1^{\mu_1} n_2^{\mu_2} \dots n_I^{\mu_I}}{p(n_1, n_2, \dots, n_I)^s}$ have been largely studied, where $(\mu_1, \mu_2, \dots, \mu_I) \in \mathbf{N}_+^I$. Mellin [5] firstly proved in 1900 that the above functions have a meromorphic continuation to whole complex plan when $\mu_i = 0$ for all indexes i , then Mahler [4] generalized the result to the case that μ_i are arbitrary positive entire numbers when the polynomial satisfies the elliptic condition in 1927. In 1987, Sargo [8] proved that the condition “ $\lim |p(n_1, n_2, \dots, n_I)| \rightarrow \infty$ when $|(n_1, n_2, \dots, n_I)| \rightarrow \infty$ ” is a necessary and sufficient condition for the above Dirichlet sequences to have a meromorphic continuation to whole complex plan. In this article, we want to prove such a condition is also necessary and sufficient for the sequences $\sum_{\underline{x} \in \mathbf{N}^n} \frac{a_{\underline{x}}}{P(\underline{x})^s}$ to have an meromorphic continuation on \mathbf{C} .

2 notations, definitions and basic properties of automatic sequences

Here we declare some notations used in this article. We denote by \underline{x} an n -tuple (x_1, x_2, \dots, x_n) we say $\underline{x} \geq \underline{y}$ (resp. $\underline{x} > \underline{y}$) if and only if $\underline{x} - \underline{y} \in \mathbf{R}_+^n$ (reps. $\underline{x} - \underline{y} \in \mathbf{R}^n$), and we have analogue definition for the symbol \leq (resp. $<$). We denote by $\underline{x}^{\underline{\mu}}$ the n -tuple $(x_1^{\mu_1}, x_2^{\mu_2}, \dots, x_n^{\mu_n})$. For a constant c , we denote by \underline{c} the tuple (c, c, \dots, c) and for two tuples \underline{x} and \underline{y} , we denote by $\langle \underline{x}, \underline{y} \rangle$ the real number $\sum_{i=1}^n x_i y_i$.

Definition Let $d \geq 2$ be an integer. A sequence $(a_n)_{n \geq 0}$ with values in the set \mathcal{A} is called d -automatic if and only if its d -kernel $\mathcal{N}_d(a)$ is finite, where the d -kernel of the sequence $(a_x)_{x \geq 0}$ is the set of subsequences defined by

$$\mathcal{N}_d(\underline{a}) = \left\{ (m_1, m_2 \dots m_n) \mapsto a_{(d^k m_1 + l_1, d^k m_2 + l_2, \dots, d^k m_n + l_n)}; k \geq 0, (0) \leq \underline{l} \leq (\underline{d^k - 1}) \right\}$$

Remark A d -automatic sequence necessarily takes finitely many values. Hence we can assume that the set \mathcal{A} is finite.

Because of the definition of q -automatic with n variables, there are some basic properties.

Theorem 1 Let $q \geq 2$ be an integer and $(a_x)_{x \geq 0}$ be a sequence with values in \mathcal{A} , the kernel of the automate. Then, the following properties are equivalent :

- (i) The sequence $(a_x)_{x \geq 0}$ is q -automatic
- (ii) There exists an integer $t \geq 1$ and a set of t sequences $\mathcal{N}' = \left\{ (a_x^1)_{x \geq 0}, \dots, (a_x^t)_{x \geq 0} \right\}$ such that
 - the sequence $(a_x^1)_{x \geq 0}$ is equal to the sequence $(a_x)_{x \geq 0}$
 - the set \mathcal{N}' is closed under the maps $(a_x)_{x \geq 0} \mapsto (a_{q\underline{x} + \underline{y}})_{x \geq 0}$ for $0 \leq \underline{y} \leq \underline{q} - 1$
- (iii) There exist an integer $t \geq 1$ and a sequence $(A_x)_{x \geq 0}$ with values in \mathcal{A}^t , that we denote as a column vector, as $(A_{1,1 \dots 1}, A_{2,1 \dots 1}, A_{1,2 \dots 1} \dots A_{1,1 \dots 2}, A_{2,2 \dots 1} \dots)^t$. There exist q^n matrices of size $t \times t$, say $M_{1,1 \dots 1}, M_{1,2 \dots 1} \dots M_{q,q \dots q}$, with the property that each row of each M_i has exactly one entry equal to 1, and the other $t - 1$ entries equal to 0, such that :
 - the first component of the vector $(A_x)_{x \geq 0}$ is the sequence $(a_x)_{x \geq 0}$
 - for each $0 \leq \underline{y} \leq \underline{q} - 1$, the equality $A_{q\underline{x} + \underline{y}} = M_{\underline{y}} A_{\underline{x}}$ holds

Proof It is a natural consequence of the finiteness of the \mathcal{N}' , the 1-index case was firstly announced in [3].

Proposition 1 Let $(a_x)_{x \geq 0}$ be a q -automatic sequence and $(b_x)_{x \geq 0}$ be a periodic sequence of period c . Then the sequence $(a_x \times b_x)_{x \geq 0}$ is also q -automatic and its q -kernel can be completed in such a way that all transition matrices of the maps $(a_x \times b_x)_{x \geq 0} \mapsto (a_{q\underline{x} + \underline{y}} \times b_{q\underline{x} + \underline{y}})_{x \geq 0}$ on the new set are independent on the choice of the values taken in the sequence $(b_x)_{x \geq 0}$.

Proof As $(a_x)_{x \geq 0}$ is a q -automatic sequence, we denote by $\mathcal{N}_a : \left\{ (a_x^{(1)})_{x \geq 0}, (a_x^{(2)})_{x \geq 0}, \dots, (a_x^{(l)})_{x \geq 0} \right\}$ its q -kernel. The sequence $(b_x)_{x \geq 0}$ is a periodic sequence, by its nature, it is also an q -automatic sequence, we denote by $\mathcal{N}_b : \left\{ (b_x^{(1)})_{x \geq 0}, (b_x^{(2)})_{x \geq 0}, \dots, (b_x^{(s)})_{x \geq 0} \right\}$ the q -kernel of $(b_x)_{x \geq 0}$. As both of the q -kernel are finite, we can conclude the set of tensor product of these two above sets is finite :

$$\mathcal{N}_{ab} : \left\{ (a_x^{(i)} \times b_x^{(j)})_{x \geq 0} \mid 0 \leq i \leq l, 0 \leq j \leq s \right\}$$

which is the q -kernel of the sequence $(a_x \times b_x)_{x \geq 0}$. We remark that there is a subjection from \mathcal{N}_b^c to \mathcal{N}_b , where \mathcal{N}_b^c is defined as

$$\left\{ (m_1, m_2, \dots, m_n) \longrightarrow (q^k m_1 + y_1, q^k m_2 + y_2, \dots, q^k m_n + y_n) \mod c; k \geq 0, (0) \leq \underline{y} \leq (\underline{d^k - 1}) \right\},$$

such a subjection is defined as

$$\mathcal{N}_b^c \rightarrow \mathcal{N}_b : (m_x)_{x \geq 0} \rightarrow (a_{m_x})_{x \geq 0}.$$

However, \mathcal{N}_b^c itself is the q -kernel of the c -periodic sequence $b^c : b^c(m_1, m_2, \dots, m_n) = (m_1, m_2, \dots, m_n) \mod c$ and its cardinal is independent on choice of the values in the sequence $(b_x)_{x \geq 0}$.

For a specific mapping $(a_x \times b_x)_{x \geq 0} \mapsto (a_{q\underline{x} + \underline{y}} \times b_{q\underline{x} + \underline{y}})_{x \geq 0}$, where $0 \leq \underline{y} < \underline{q}$, on the set \mathcal{N}_{ab} , it can be completed to $(a_x \times b_x^c)_{x \geq 0} \mapsto (a_{q\underline{x} + \underline{y}} \times b_{q\underline{x} + \underline{y}}^c)_{x \geq 0}$, where $0 \leq \underline{x} < \underline{q}$ and this completion can be represented uniquely by the tensor product of two mappings $(a_x)_{x \geq 0} \mapsto (a_{q\underline{x} + \underline{y}})_{x \geq 0}$ and $(b_x^c)_{x \geq 0} \mapsto (b_{q\underline{x} + \underline{y}}^c)_{x \geq 0}$, as each has a unique transition matrix on their own q -kernel, the tensor product of them is also unique on \mathcal{N}_{ab}^c and all these transition matrices and completion of the q -kernel are independent on the choice of values in the sequence $(b_x)_{x \geq 0}$

Let us consider the Dirichlet series $f(s) = \sum_{\underline{x} \in \mathbf{N}^n / \underline{0}} \frac{a_{\underline{x}}}{p(\underline{x})^s}$, Where $a_{\underline{x}}$ is a q -automate, a necessary condition of the convergence of such series is that $|p(\underline{x})| \rightarrow \infty$ when $|\underline{x}| \rightarrow \infty$ (meaning $p(\underline{x}) \rightarrow +\infty$ or $p(\underline{x}) \rightarrow -\infty$ when $|\underline{x}| \rightarrow \infty$), here we want to show this is a sufficient condition.

Definition Let $p(\underline{x}) = \sum_{\underline{\alpha}} a_{\underline{\alpha}} x^{\underline{\alpha}}$ be an n -variable polynomial. We denote by $\text{supp}(p)$ the set of multi-index $\underline{\alpha}$ such that $a_{\underline{\alpha}}$ is non zero. and $\mathcal{E}(p)$ the convex envelope of the subset of \mathbf{R}^n

$$\{\underline{\alpha} - \underline{\beta} / \underline{\alpha} \in \text{supp}(p), \underline{\beta} \in \mathbf{R}^n\}$$

We say a monomial $\underline{x}^{\underline{\alpha}}$ is extremal if $\underline{\alpha}$ is an extremal point of $\mathcal{E}(p)$. If p does not have any extremal point, then it has a biggest monomial such that $p(\underline{x})$ is written as $p(\underline{x}) = a_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} + \sum_{\underline{\beta} < \underline{\alpha}} a_{\underline{\beta}} \underline{x}^{\underline{\beta}}$.

3 proof of meromorphic continuation

In this section we prove the main result read as below :

Theorem 2 Let p be a polynomial of n variables that $|p(\underline{x})| \rightarrow \infty$ when $|\underline{x}| \rightarrow \infty$, then for a given n -tuple $\underline{\mu}$, if there exists an $l \in \mathbf{N}$ such that $\prod_{i=1}^n x_i^{\mu_i} \in \mathcal{E}(p^l)$ then $\sum_{(\underline{x}) \in \mathbf{N}^n / \underline{0}} \frac{a_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s}$ admits an abscissa of convergence σ such that it converge absolutely on the semi plan $\mathcal{R}_e(s) > \sigma$ and has a meromorphic continuation on whole complex plan. What is more, the number of poles of this function (if any) is finite and located on a finite set.

This result will be declared by proving several lemmas successively :

Lemma 1 Let $a_{\underline{x}}$ be an q -automate, and $p(\underline{x}) = \sum_{\underline{\alpha}} m_{\underline{\alpha}} x^{\underline{\alpha}}$ be a n -variable homogeneous polynomial, of degree d , with positive coefficients such that $|p(\underline{x})| \rightarrow \infty$ when $|\underline{x}| \rightarrow \infty$, let $\underline{\mu} \in \mathbf{N}_+^n$ be a multi-index and $l \in \mathbf{N}$ such that $\prod_{i=1}^n x_i^{\mu_i} \in \mathcal{E}(p^l)$, for any $\underline{0} \leq \underline{\beta} \leq \underline{q}$, define $p_{\underline{\beta}}(\underline{x}) = q^{-n}(p(q\underline{x} + \underline{\beta}) - p(q\underline{x}))$, then for any $k \in \mathbf{N}$, the function $f_{k, \underline{\beta}, \underline{\mu}} : s \rightarrow \sum_{(\underline{x}) \in \mathbf{N}^n / \underline{0}} \frac{a_{\underline{x}} p_{\underline{\beta}}(\underline{x})^k \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^{s+k}}$ admits an abscissa of convergence $\sigma_{k, \underline{\beta}, \underline{\mu}}$ such that $F_{k, \underline{\beta}, \underline{\mu}}$ converge to an holomorphic function on the right half plan $\mathcal{R}_e(s) > \sigma_{k, \underline{\beta}, \underline{\mu}}$.

Proof We firstly prove that $f_{0, \underline{0}, \underline{0}}(s)$ converge when $\mathcal{R}_e(s) > n$.

$$\begin{aligned} |f_{0, \underline{0}, \underline{0}}(s)| &= \left| \sum_{(\underline{x}) \in \mathbf{N}^n / \underline{0}} \frac{a_{\underline{x}}}{p(\underline{x})^s} \right| \leq \sum_{(\underline{x}) \in \mathbf{N}^n / \underline{0}} \frac{|a_{\underline{x}}|}{p(\underline{x})^{\mathcal{R}_e(s)}} \leq \sum_{(\underline{x}) \in \mathbf{N}^n / \underline{0}} \frac{|a_{\underline{x}}|}{(\min(m_{\alpha}) < \underline{x}, \underline{1} >)^{\mathcal{R}_e(s)}} \\ &\leq \frac{\max(|a_{\underline{x}}|)}{(\min(m_{\alpha})^{\mathcal{R}_e(s)})} \left(\sum_{< \underline{x}, \underline{1} > < n} \frac{1}{< \underline{x}, \underline{1} >^{\mathcal{R}_e(s)}} + \sum_{< \underline{x}, \underline{1} > \geq n} \frac{1}{< \underline{x}, \underline{1} >^{\mathcal{R}_e(s)}} \right) \\ &\leq \frac{\max(|a_{\underline{x}}|)}{(\min(m_{\alpha})^{\mathcal{R}_e(s)})} \left(\sum_{< \underline{x}, \underline{1} > < n} \frac{1}{< \underline{x}, \underline{1} >^{\mathcal{R}_e(s)}} + \sum_{m \geq n} \frac{\binom{m+n-1}{n-1}}{m^{\mathcal{R}_e(s)}} \right) \\ &\leq \frac{\max(|a_{\underline{x}}|)}{(\min(m_{\alpha})^{\mathcal{R}_e(s)})} \left(\sum_{< \underline{x}, \underline{1} > < n} \frac{1}{< \underline{x}, \underline{1} >^{\mathcal{R}_e(s)}} + \sum_{m \geq n} \frac{m^{n-1}}{m^{\mathcal{R}_e(s)}} \right) \\ &\leq \frac{\max(|a_{\underline{x}}|)}{(\min(m_{\alpha})^{\mathcal{R}_e(s)})} \left(\sum_{< \underline{x}, \underline{1} > < n} \frac{1}{< \underline{x}, \underline{1} >^{\mathcal{R}_e(s)}} + \sum_{m \geq n} \frac{1}{m^{\mathcal{R}_e(s)+1-n}} \right) \end{aligned} \tag{1}$$

the summation $\sum_{m \geq n} \frac{1}{m^{\mathcal{R}_e(s)+1-n}}$ exists and is bounded when $\mathcal{R}_e(s) > n$.

For any $\underline{0} \leq \underline{\beta} < \underline{p}$, and any multi-index $\underline{k} \in \text{supp}(p)$, we remark that $\prod_{i=0}^n (x_i + \beta_i)^{k_i} = \sum_{l \leq \underline{k}} C_l \prod_{i=0}^n (x_i)^{k_i}$ where $C_l < p^n$, which shows all monomials of the polynomial $p_{\underline{\beta}}$, of degree $d-1$, can be majored by one of p 's. This effect shows that, $\frac{p_{\underline{\beta}}(\underline{x})}{p(\underline{x})} \rightarrow 0$ when $|\underline{x}| \rightarrow \infty$. There is an $C_0 \in \mathbf{R}^+$ such that for all $< \underline{x}, \underline{1} > \geq C_0$, $|\frac{p_{\underline{\beta}}(\underline{x})}{p(\underline{x})}| < 1/2$. For any $k > 0$, take $C_1 = \max(n, C_0)$

$$\sum_{(\underline{x}) \in \mathbf{N}^n / \underline{0}} \left| \frac{a_{\underline{x}} p_{\underline{\beta}}^k(\underline{x}) \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^{s+k}} \right| \leq \sum_{(\underline{x}) \in \mathbf{N}^n / \underline{0}} \frac{a_{\underline{x}}}{p^{\mathcal{R}_e(s)-l}(\underline{x})} \frac{p_{\underline{\beta}}^k(\underline{x})}{p^k(\underline{x})} \leq \sum_{< \underline{x}, \underline{1} > < C_1} \frac{a_{\underline{x}}}{p^{\mathcal{R}_e(s)-l}(\underline{x})} + (1/2)^k \sum_{< \underline{x}, \underline{1} > \geq C_1} \frac{a_{\underline{x}}}{p^{\mathcal{R}_e(s)-l}(\underline{x})}$$

With K a constant in \mathbf{R}^+ , the above function converge to an holomorphic function on the half plan $\mathcal{R}_e(s) > n + l$. what is more, for all $b > k$, $\sum_{\langle \underline{x}, \underline{1} \rangle \geq x_1} \left| \frac{a_{\underline{x}} p_{\beta}^b(\underline{x})}{p^{s-l+b}(\underline{x})} \right|$ is bounded on this half plan.

Lemma 2 *With the same notations and hypotheses as above, The function $F : s \longrightarrow \sum_{(\underline{x}) \in \mathbf{N}^n / \underline{0}} \frac{a_{\underline{x}}}{p(\underline{x})^s}$ admit a meromorphic continuation on the whole complex plan.*

Proof In this proof, we consider the q -automatic sequence $(a_{\underline{x}})_{\underline{x} \geq \underline{0}}$ as itself multiplied by an constant sequence $(b_{\underline{x}})_{\underline{x} \geq \underline{0}} = 1$, which is a q -automatic sequence. Because of the proposition 1, the q -kernel of this sequence admits a completion, we can define a sequence of vector $(A_{\underline{x}})_{\underline{x} \geq \underline{0}}$ and the matrices of transition on this completion as we do in the theory 1.

For any $\underline{\mu} \in \mathbf{N}_+^n$, considering the function $F_{\underline{\mu}}$ defined as $F_{\underline{\mu}}(s) = \sum_{(\underline{x}) \in \mathbf{N}^n / (\underline{0})} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s}$, where $A_{\underline{x}}$ is defined as above, and an constant $N_0 \in \mathbf{N}$ such that $C_1 < N_0 np$, where C_1 is defined as in the previous lemma.

$$\begin{aligned} F_{\underline{\mu}}(s) &= \sum_{(\underline{x}) \in \mathbf{N}^n / (\underline{0})} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} = \sum_{(\underline{x}) < (N_0 q)} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} + \sum_{(\underline{y}) < (q)} \sum_{(\underline{z}) \in \mathbf{N}^n / \{\underline{t} < N_0\}} \frac{A_{q\underline{z} + \underline{y}} \prod_{i=1}^n (qz_i + y_i)^{\mu_i}}{p^s(q\underline{z} + \underline{y})} \\ &= \sum_{(\underline{x}) < (N_0 q)} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} + \sum_{(\underline{y}) < (q)} \sum_{(\underline{z}) \in \mathbf{N}^n / \{\underline{t} < N_0\}} \frac{A_{q\underline{z} + \underline{y}} \prod_{i=1}^n (qz_i)^{\mu_i}}{p^s(q\underline{z} + \underline{y})} + \sum_{(\underline{\psi}) < (\underline{\mu})} \sum_{(\underline{y}) < (q)} \sum_{(\underline{z}) \in \mathbf{N}^n / \{\underline{t} < N_0\}} \frac{A_{q\underline{z} + \underline{y}} C_{\underline{\psi}, \underline{y}} \prod_{i=1}^n (qz_i + y_i)^{\psi_i}}{p^s(q\underline{z} + \underline{y})} \end{aligned} \quad (2)$$

Where $C_{\underline{\psi}, \underline{y}}$ is a periodic coefficient in function of \underline{x} , we denote by $Res_{\underline{\mu}}(s)$ the therm

$$Res_{\underline{\mu}}(s) = \sum_{(\underline{\psi}) < (\underline{\mu})} \sum_{(\underline{y}) < (q)} \sum_{(\underline{z}) \in \mathbf{N}^n / \{\underline{t} < N_0\}} \frac{A_{q\underline{z} + \underline{y}} C_{\underline{\psi}, \underline{y}} \prod_{i=1}^n (qz_i + y_i)^{\psi_i}}{p^s(q\underline{z} + \underline{y})}.$$

We remark that all sequences $a_{q\underline{z} + \underline{y}} C_{\underline{\psi}, \underline{y}}$ are in the form of a production of a specific q -automatic sequence by a q -periodic one, because of the proposition 1, such sequences admit a unique completion, the same one as $a_{\underline{x}}$ has, and the transition matrices on this completion is invariant on function of the q -periodic sequences $C_{\underline{\psi}, \underline{y}}$.

$$\begin{aligned} F_{\underline{\mu}}(s) &= \sum_{(\underline{x}) \in \mathbf{N}^n / (\underline{0})} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} = \sum_{(\underline{x}) < (N_0 q)} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} + \sum_{(\underline{y}) < (q)} \sum_{(\underline{z}) \in \mathbf{N}^n / \{\underline{t} < N_0\}} \frac{A_{q\underline{z} + \underline{y}} \prod_{i=1}^n (qz_i + y_i)^{\mu_i}}{p^s(q\underline{z} + \underline{y})} \\ &= \sum_{(\underline{x}) < (N_0 q)} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} + \sum_{(\underline{y}) < (q)} \sum_{(\underline{z}) \in \mathbf{N}^n / \{\underline{t} < N_0\}} \frac{A_{q\underline{z} + \underline{y}} \prod_{i=1}^n (qz_i)^{\mu_i}}{p^s(q\underline{z} + \underline{y})} + Res_{\underline{\mu}}(s) \\ &= \sum_{(\underline{x}) < (N_0 q)} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} + \sum_{(\underline{y}) < (q)} M_{\underline{y}} \sum_{(\underline{z}) \in \mathbf{N}^n / \{\underline{t} < N_0\}} \frac{A_{q\underline{z}} \prod_{i=1}^n (qz_i)^{\mu_i}}{p^s(q\underline{z})} \frac{1}{(1 + \frac{p_{\underline{y}}(\underline{z})}{p(\underline{z})})^s} + Res_{\underline{\mu}}(s) \\ &= \sum_{(\underline{x}) < (N_0 q)} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} + \sum_{(\underline{y}) < (q)} M_{\underline{y}} \sum_{(\underline{z}) \in \mathbf{N}^n / \{\underline{t} < N_0\}} \frac{A_{\underline{z}} \prod_{i=1}^n (qz_i)^{\mu_i}}{p^s(q\underline{z})} \sum_{k \geq 0} \binom{-s+k}{k} \left(\frac{-p_{\underline{y}}(\underline{z})}{p(\underline{z})} \right)^k + Res_{\underline{\mu}}(s) \\ &= \sum_{(\underline{x}) < (N_0 q)} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} + q^{\langle \underline{\mu}, \underline{1} \rangle - ns} \sum_{(\underline{y}) < (q)} M_{\underline{y}} \sum_{k \geq 0} \binom{-s+k}{k} \sum_{(\underline{z}) \in \mathbf{N}^n / \{\underline{t} < N_0\}} \frac{A_{\underline{z}} \prod_{i=1}^n (z_i)^{\mu_i} (-p_{\underline{y}}(\underline{z}))^k}{(p(\underline{z}))^{s+k}} + Res_{\underline{\mu}}(s) \end{aligned} \quad (3)$$

The above equation gives :

$$\begin{aligned} (Id - q^{\langle \underline{\mu}, \underline{1} \rangle - ns} \sum_{(\underline{y}) < (q)} M_{\underline{y}}) F_{\underline{\mu}}(s) &= q^{\langle \underline{\mu}, \underline{1} \rangle - ns} \sum_{(\underline{y}) < (q)} M_{\underline{y}} \sum_{k \geq 1} \binom{-s+k}{k} \sum_{(\underline{z}) \in \mathbf{N}^n / \{\underline{t} < N_0\}} \frac{A_{\underline{z}} \prod_{i=1}^n (z_i)^{\mu_i} (-p_{\underline{y}}(\underline{z}))^k}{(p(\underline{z}))^{s+k}} \\ &\quad + \sum_{(\underline{x}) < (N_0 q)} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} + Res_{\underline{\mu}}(s) \end{aligned} \quad (4)$$

By multiplying $com^t(Id - q^{<\underline{\mu}, \underline{1}>-ns} \sum_{(\underline{y}) < (\underline{q})} M_{\underline{y}})$ on both side, we have :

$$\begin{aligned} det(Id - q^{<\underline{\mu}, \underline{1}>-ns} \sum_{(\underline{y}) < (\underline{q})} M_{\underline{y}}) F_{\underline{\mu}}(s) &= com^t(Id - q^{<\underline{\mu}, \underline{1}>-ns} \sum_{(\underline{y}) < (\underline{q})} M_{\underline{y}}) \left(\sum_{(\underline{x}) < (N_0 \underline{q})} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} + Res_{\underline{\mu}}(s) \right. \\ &\quad \left. + q^{<\underline{\mu}, \underline{1}>-ns} \sum_{(\underline{y}) < (\underline{q})} M_{\underline{y}} \sum_{k \geq 1} \binom{-s+k}{k} \sum_{(\underline{z}) \in \mathbf{N}^n / \{\underline{t} < N_0\}} \frac{A_{\underline{z}} \prod_{i=1}^n (z_i)^{\mu_i} (-p_{\underline{y}}(\underline{z}))^k}{(p(\underline{z}))^{s+k}} \right) \end{aligned} \quad (5)$$

The above formula shows that $F_{\underline{0}}$ has a meromorphic continuation on the half plan on $\mathcal{R}_e(s) > n - 1$. Thanks to the previous lemma, we have for any $\underline{\mu} \in \mathbf{N}_+^n$, such that $\prod_{i=1}^n x_i^{\mu_i} \in \mathcal{E}(p)$, the function $F_{\underline{\mu}}(s)$ converges and is bounded when $\mathcal{R}_e(s) > n + 1$. The equation 5 shows that, all of these functions $F_{\underline{\mu}}$ have a meromorphic continuation on $\mathcal{R}_e(s) > n$. In particular, for any $\underline{0} \leq \underline{\beta} < \underline{q}$, the functions $s \rightarrow \sum_{(\underline{x}) \in \mathbf{N}^n / (\underline{0})} \frac{A_{\underline{x}} p_{\underline{\beta}}(\underline{x})}{p(\underline{x})^s}$ have a meromorphic continuation on $\mathcal{R}_e(s) > n$, this effect proves that $F_{\underline{0}}$ has a meromorphic continuation on the half plan on $\mathcal{R}_e(s) > n - 2$. We do this successively to prove the meromorphic continuation on $\mathcal{R}_e(s) > n - 3 \dots$. To conclusion, The function $F_{\underline{0}}$ has a meromorphic continuation on whole complex plan.

What is more, the poles of such function can only locate at the zeros of the function $s \rightarrow det(Id - q^{<\underline{\mu}, \underline{1}>-ns} \sum_{(\underline{y}) < (\underline{q})} M_{\underline{y}})$ for an arbitrary $\underline{\mu} \in \mathbf{N}_+^n$, so we conclude that all poles of function $F(s)$ are located in the set

$$s = \frac{1}{n} \left(\frac{\log \lambda}{\log q} + \frac{2ik\pi}{\log q} - l \right)$$

with λ any eigenvalue of the matrix $\sum_{(\underline{y}) < (\underline{q})} M_{\underline{y}}$, $k \in \mathbf{Z}$, $l \in \mathbf{Z}$ and \log is defined as complex logarithm.

Remark The previous lemma proves indeed that for all monomial $\prod_{i=1}^n x_i^{\mu_i} \in \mathcal{E}(p^l)$ for some $l \in \mathbf{N}$ the function $s \rightarrow \sum_{(\underline{x}) \in \mathbf{N}^n / (\underline{0})} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s}$ admits a meromorphic continuation on whole complex plan.

Lemma 3 Let p be an n -variable polynomial, of degree d , such that $|p(\underline{x})| \rightarrow \infty$ when $|\underline{x}| \rightarrow \infty$, and all its terms are extreme, let $a_{\underline{x}}$ be an automate, then $G : s \rightarrow \sum_{(\underline{x}) \in \mathbf{N}^n / (\underline{0})} \frac{a_{\underline{x}}}{p^s(\underline{x})}$ is meromorphic.

Proof The hypotheses $|p(\underline{x})| \rightarrow \infty$ when $|\underline{x}| \rightarrow \infty$ shows that all coefficients of this polynomial are positive or negative at the same time. Without loss of generality, we suppose that all its coefficient are positive. We build another sequence $b_{\underline{y}}$ of $n + 1$ -variable in such a way that $b_{\underline{y}} = 0$ if $y_{n+1} \neq 1$ and $b_{\underline{y}} = a_{y_1, y_2, \dots, y_n}$ if $y_{n+1} = 1$. we remark that the sequence $b_{\underline{y}}$ is also automatic because its q -kernel is finite. And we complete the function p to an $n + 1$ -variable polynomial g of degree d in such a way that we multiply x_{n+1} to all terms whose degrees are smaller then d until they all have the same degree d . so we have

$$G(s) = \sum_{(\underline{x}) \in \mathbf{N}^n / (\underline{0})} \frac{a_{\underline{x}}}{p^s(\underline{x})} = \sum_{(\underline{y}) \in \mathbf{N}^{n+1} / (\underline{0})} \frac{b_{\underline{y}}}{g^s(\underline{y})}$$

The previous lemma proves that this function has a meromorphic continuation to whole complex plan.

Remark We have the same remark as before : for any monomial $\prod_{i=1}^n x_i^{\mu_i} \in \mathcal{E}(g^l)$ for some $l \in \mathbf{N}$ the function $s \rightarrow \sum_{(\underline{x}) \in \mathbf{N}^n / (\underline{0})} \frac{B_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s}$ admits a meromorphic continuation on whole complex plan.

proof of the theorem 2 Let us write the polynomial $p(\underline{x})$ in the form $p(\underline{x}) = p_0(\underline{x}) + Res(\underline{x})$ where $p_0(\underline{x})$ is the summation of all extreme monomials of $p(\underline{x})$, it is easy to see $\frac{Res(\underline{x})}{p_0(\underline{x})} = \mathcal{O}(x^{-1})$. So there exists $x_0 \in \mathbf{N}_+$ such that for all $|\underline{x}| > x_0$, $|\frac{Res(\underline{x})}{p_0(\underline{x})}| < 1$. For any given semi plan $\{s | \mathcal{R}_e(s) > m, m \in \mathbf{R}\}$, take an integer $s_0 > 2 - dm + < \underline{\mu}, \underline{1} >$ where d denoted the degree of the polynomial. we can conclude that $\sum_{|\underline{x}| > x_0} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{(p_0(\underline{x}))^s} \sum_{k=s_0+1}^{\infty} \binom{-s+k}{k} \frac{Res^k(\underline{x})}{p_0^k(\underline{x})} = \mathcal{O}(|s|^{s_0})$. So we have the equivalence as below :

$$\begin{aligned}
\sum_{(\underline{x}) \in \mathbf{N}_+^n / (\mathbb{Q})} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} &= \sum_{|\underline{x}| \leq x_0} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} + \sum_{|\underline{x}| > x_0} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{(p_0(\underline{x}))^s} \frac{1}{(1 + \frac{Res(\underline{x})}{p_0(\underline{x})})^s} \\
&= \sum_{|\underline{x}| \leq x_0} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} + \sum_{|\underline{x}| > x_0} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{(p_0(\underline{x}))^s} \sum_{k=0}^{s_0} \binom{-s+k}{k} \frac{Res^k(\underline{x})}{p_0^k(\underline{x})} + \mathcal{O}(|s|^{s_0})
\end{aligned} \tag{6}$$

For each $k \in \mathbf{N}$,

$$\sum_{|\underline{x}| > x_0} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{(p_0(\underline{x}))^s} \frac{Res^k(\underline{x})}{p_0^k(\underline{x})} = \sum_{j \leq k} \sum_{|\underline{x}| > x_0} \frac{A_{\underline{x}} \mathcal{C}_{\underline{x}}}{(p_0(\underline{x}))^{s+k}} \prod_{i=1}^n x_i^{j_i} \tag{7}$$

Where $\mathcal{C}_{\underline{x}}$ are constants to \underline{x} , and $s \rightarrow \sum_{|\underline{x}| > x_0} \frac{A_{\underline{x}} \mathcal{C}_{\underline{x}}}{(p_0(\underline{x}))^{s+k}} \prod_{i=1}^n x_i^{j_i}$ are meromorphic functions because of the previous lemma, in effect, if we complete the polynomial p_0 to an $n+1$ -variable polynomial g , then all term of the form $\prod_{i=1}^n x_i^{\mu_i}$ are in the convex set $\mathcal{E}(g^l)$ for some l , and we use the remark of lemma 3. As there are finitely many of meromorphic function in (6), we can conclude that for every $k > 0$, $s \rightarrow \sum_{|\underline{x}| > x_0} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{(p_0(\underline{x}))^s} Res^k(\underline{x})$ is meromorphic. This proves that for an arbitrary $s_0 \in \mathbf{R}$ the function $f(s)$ is a finite summation of meromorphic continuous functions on the half plan $\mathcal{R}_e(s) > s_0$, so itself is meromorphic continuous on this half plan. To conclusion, the function $s \rightarrow \sum_{(\underline{x}) \in \mathbf{N}^n / (\mathbb{Q})} \frac{a_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s}$ is meromorphic on whole complex plane.

Proposition 2 Let $f(s) = \sum_{(\underline{x}) \in \mathbf{N}^n / (\mathbb{Q})} \frac{a_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s}$ be the function defined as in theorem 2. Let s_0 be its first pole on the axis of real number counting from plus infinity to minus infinity. Then the function $H(s)$ has a simple pole on this point.

Proof We recall a classical result of matrices (see [6]) : Let B be a matrix of size $t \times t$ over any commutative field, $p_B(X)$ be its characteristic polynomial, and $\pi_B(X)$ be its monic minimal polynomial. Let $\Delta(X)$ be the monic gcd of the entries of (the transpose of) the comatrix of the matrix $(B - XI)$, then :

$$p_B(X) = (-1)^t \pi_B(X) \Delta(X)$$

Let us denote by B the matrix $(nq)^{-1} \sum_{(\underline{y}) < (\underline{q})} M_{\underline{y}}$ and by T its size. By devising $\Delta(q^{n(s-1)+<\underline{\mu}, \underline{1}>})$ on both sides of the formula 5, we get :

$$\begin{aligned}
\pi_B(q^{n(s-1)+<\underline{\mu}, \underline{1}>}) H(s) &= \frac{com^t(q^{n(s-1)+<\underline{\mu}, \underline{1}>}) Id - \sum_{(\underline{y}) < (\underline{q})} M_{\underline{y}}}{\Delta(q^{n(s-1)+<\underline{\mu}, \underline{1}>})} \left(\sum_{(\underline{x}) < (\underline{N_0 q})} \frac{A_{\underline{x}} \prod_{i=1}^n x_i^{\mu_i}}{p(\underline{x})^s} + Res_{\mu}(s) \right) \\
&+ q^{<\underline{\mu}, \underline{1}>-ns} \sum_{(\underline{y}) < (\underline{q})} M_{\underline{y}} \sum_{k \geq 1} \binom{-s+k}{k} \sum_{(\underline{z}) \in \mathbf{N}^n / \{\underline{t} < \underline{N_0}\}} \frac{A_{\underline{z}} \prod_{i=1}^n (z_i)^{\mu_i} (-p_{\underline{y}}(\underline{z}))^k}{(p(\underline{z}))^{s+k}}
\end{aligned} \tag{8}$$

The right hand side of above function is holomorphic when $\mathcal{R}_e(s) > s_0$. As s_0 is the first pole of $H(s)$ on real axis counting from plus infinity, it is a zero point of function $\pi_B(q^{n(s-1)+<\underline{\mu}, \underline{1}>})$ associated to the eigenvalue 1 of the matrix B . On the other hand, as B is a stochastic matrix, it has a simple root on 1, so the function $\pi_B(q^{n(s-1)+<\underline{\mu}, \underline{1}>})$ has a simple root on s_0 which conclude the proposition.

4 Infinite products

Let $P(x) = \sum_{i=0}^d a_i x^i$ be a polynomial which does not have zeros on \mathbb{Q} and $\tilde{P}(x)$ be another such that $\tilde{P}(x) = \sum_{i=0}^{d-1} -\frac{a_i}{a_d} x^{n-i}$, by definition, we have $P(x) = a_d x^d - a_d x^d \tilde{P}(\frac{1}{x})$. Let us define $c_i = \frac{a_i}{a_d}$ for all $i = 0, 1, \dots, d-1$.

In this section we consider two Dirichlet series generated by 1-index automatic sequences :

$$f(s) = \sum_{n=0}^{\infty} \frac{(\zeta)^{S_q(n)}}{(P(n+1))^s}$$

$$g(s) = \sum_{n=1}^{\infty} \frac{(\zeta)^{S_q(n)}}{(P(n))^s}$$

Where q and r are two integers satisfying $2 \leq r \leq q$ and r divides q , ζ is a r -th root of unity, such that $\zeta \neq 1$, $s_q(n)$ is the sum of digits of n in the q -ary expansion satisfying $s_q(0) = 0$ and $s_q(qn+a) = s_q(n)+a$ for $0 \leq a \leq q-1$.

Let us denote by :

$$\begin{aligned}\phi(s) &= \sum_{n=0}^{\infty} \frac{(\zeta)^{S_q(n)}}{(n+1)^s} \\ \psi(s) &= \sum_{n=1}^{\infty} \frac{(\zeta)^{S_q(n)}}{(n)^s}\end{aligned}$$

It is proved in [2] that ϕ and ψ have holomorphic continuations to the whole complex plane, and $\psi(s)(q^s - 1) = \phi(s)(\zeta q^s - 1)$ for all $s \in \mathbb{C}$

Proposition 3 *f and g also have holomorphic continuations to the whole complex plane if all their coefficients c_i satisfy $\max |c_i| < \frac{1}{d}$.*

Proof We firstly remark that the hypothesis of $\max |c_i| < \frac{1}{d}$ induces the fact $|\tilde{P}(\frac{1}{n+1})| < 1$ for any $n \in \mathbf{N}_+$. Indeed, $|\tilde{P}(\frac{1}{n+1})| = |\sum_{i=0}^{d-1} -\frac{a_i}{a_d}(n+1)^{d-i}| \leq \sum_{i=0}^{d-1} |\frac{a_i}{a_d}| < 1$.

$$\begin{aligned}f(s) &= \sum_{n=0}^{\infty} \frac{(\zeta)^{S_q(n)}}{(P(n+1))^s} \\ &= a_d^{-s} \sum_{n=0}^{\infty} \frac{(\zeta)^{S_q(n)}}{(n+1)^{ds} (1 - \tilde{P}(\frac{1}{n+1}))^s} \\ &= a_d^{-s} \sum_{n=0}^{\infty} \frac{(\zeta)^{S_q(n)}}{(n+1)^{ds}} \sum_{k=0}^{\infty} \binom{s+k-1}{k} \tilde{P}(\frac{1}{n+1})^k \\ &= a_d^{-s} \sum_{n=0}^{\infty} \frac{(\zeta)^{S_q(n)}}{(n+1)^{ds}} \sum_{k=0}^{\infty} \binom{s+k-1}{k} \sum_{l=k}^{dk} m_{k,l} (n+1)^{-l} \\ &= a_d^{-s} \sum_{k=0}^{\infty} \binom{s+k-1}{k} \sum_{l=k}^{dk} m_{k,l} \sum_{n=0}^{\infty} \frac{(\zeta)^{S_q(n)}}{(n+1)^{ds+l}} \\ &= a_d^{-s} \sum_{k=0}^{\infty} \binom{s+k-1}{k} \sum_{l=k}^{dk} m_{k,l} \phi(ds+l) \\ &= a_d^{-s} \phi(ds) + a_d^s \sum_{k=1}^{\infty} \binom{s+k-1}{k} \sum_{l=k}^{dk} m_{k,l} \phi(ds+l)\end{aligned} \tag{9}$$

Where $m_{k,l} = \sum_{M_{k,l} \in \mathcal{P}(\{c_i | 1 \leq i \leq n-1\})} \prod_{c_i \in M_{k,l}} c_i$ and $M_{k,l}$ are sets of k elements included in $\{c_i | 1 \leq i \leq n-1\}$ and the sum of indices of its elements equals l . The hypothesis $m = \max |c_i| < \frac{1}{d}$ shows that $|\sum_{l=k}^{dk-k} m_{k,l}| \leq (md)^k < 1$, so the left part of (9) converge because $\phi(s)$ is bounded for large $|s|$, which proves the holomorphic continuation of f on the whole complex plane.

It is easy to check $f(0) = 0$, so we have :

$$\begin{aligned}
f'(0) &= d\phi'(0) + a_d^{-s} \sum_{k=1}^{\infty} \lim_{s \rightarrow 0} \frac{1}{s} \binom{s+k-1}{k} \sum_{l=k}^{dk} m_{k,l} \phi(ds+l) \\
&= -d \log q / (\zeta - 1) + \sum_{k=1}^{\infty} k^{-1} \sum_{l=k}^{dk} m_{k,l} \phi(l) \\
&= -d \log q / (\zeta - 1) + \sum_{k=1}^{\infty} k^{-1} \sum_{l=k}^{dk} m_{k,l} \sum_{n=0}^{\infty} \frac{(\zeta)^{S_q(n)}}{(n+1)^l} \\
&= -d \log q / (\zeta - 1) + \sum_{n=0}^{\infty} (\zeta)^{S_q(n)} \sum_{k=1}^{\infty} k^{-1} \sum_{l=k}^{dk} m_{k,l} \frac{1}{(n+1)^l} \\
&= -d \log q / (\zeta - 1) + \sum_{n=0}^{\infty} (\zeta)^{S_q(n)} \sum_{k=1}^{\infty} k^{-1} \tilde{P}^k \left(\frac{1}{n+1} \right) \\
&= -d \log q / (\zeta - 1) + \sum_{n=0}^{\infty} (\zeta)^{S_q(n)} \log \left(1 - \tilde{P} \left(\frac{1}{n+1} \right) \right)
\end{aligned} \tag{10}$$

on the other hand, one has for all s , $\psi(s)(q^s - 1) = \phi(s)(\zeta q^s - 1)$

$$\begin{aligned}
f'(0) &= d\phi'(0) + a_d^{-s} \sum_{k=1}^{\infty} \lim_{s \rightarrow 0} \frac{1}{s} \binom{s+k-1}{k} \sum_{l=k}^{dk} m_{k,l} \phi(ds+l) \\
&= -d \log q / (\zeta - 1) + \sum_{k=1}^{\infty} k^{-1} \sum_{l=k}^{dk} m_{k,l} \psi(l) (q^l - 1) / (\zeta q^l - 1) \\
&= -d \log q / (\zeta - 1) + \zeta^{-1} \sum_{k=1}^{\infty} k^{-1} \sum_{l=k}^{dk} m_{k,l} \psi(l) + (\zeta^{-1} - 1) \sum_{k=1}^{\infty} k^{-1} \sum_{l=k}^{dk} m_{k,l} \psi(l) / (\zeta q^l - 1)
\end{aligned} \tag{11}$$

by the same method as above, we can deduce

$$\sum_{k=1}^{\infty} k^{-1} \sum_{l=k}^{dk} m_{k,l} \psi(l) = \sum_{n=1}^{\infty} (\zeta)^{S_q(n)} \log \left(1 - \tilde{P} \left(\frac{1}{n} \right) \right) \tag{12}$$

and

$$\begin{aligned}
\sum_{k=1}^{\infty} k^{-1} \sum_{l=k}^{dk} m_{k,l} \psi(l) / (\zeta q^l - 1) &= \sum_{k=1}^{\infty} k^{-1} \sum_{l=k}^{dk} m_{k,l} \sum_{n=1}^{\infty} \frac{(\zeta)^{S_q(n)}}{(n)^l} \sum_{r=1}^{\infty} (\zeta q^l)^{-r} \\
&= \sum_{r=1}^{\infty} (\zeta)^{-r} \sum_{n=1}^{\infty} (\zeta)^{S_q(n)} \sum_{k=1}^{\infty} k^{-1} \sum_{l=k}^{dk} m_{k,l} \frac{1}{(nq^r)^l} \\
&= \sum_{r=1}^{\infty} (\zeta)^{-r} \sum_{n=1}^{\infty} (\zeta)^{S_q(n)} \log \left(1 - \tilde{P} \left(\frac{1}{nq^r} \right) \right)
\end{aligned} \tag{13}$$

As a consequence,

$$\sum_{n=0}^{\infty} (\zeta)^{S_q(n)} \log \left(1 - \tilde{P} \left(\frac{1}{n+1} \right) \right) = \zeta^{-1} \sum_{n=1}^{\infty} (\zeta)^{S_q(n)} \log \left(1 - \tilde{P} \left(\frac{1}{n} \right) \right) + (\zeta^{-1} - 1) \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} (\zeta)^{S_q(n)-r} \log \left(1 - \tilde{P} \left(\frac{1}{nq^r} \right) \right) \tag{14}$$

Proposition 4 *We have the equality*

$$\prod_{n=0}^{\infty} \left(1 - \tilde{P} \left(\frac{1}{n+1} \right) \right)^{\zeta^{S_{q^n}}} \times \prod_{n=1}^{\infty} \left(1 - \tilde{P} \left(\frac{1}{n} \right) \right)^{-\zeta^{S_{q^n}-1}} \times \left(\prod_{r=1}^{\infty} \prod_{n=1}^{\infty} \left(1 - \tilde{P} \left(\frac{1}{nq^r} \right) \right)^{\zeta^{S_{q^n}-r}} \right)^{1-\zeta^{-1}} = 1 \tag{15}$$

Let us consider an example as follows : for $q = 2$, then $\zeta = -1$, if we take $p(x) = 5x^2 - x - 1$ then $\tilde{p}(x) = 0.2x^2 + 0.2x$. We compute separately the values of these equations : $A(N) = \prod_{n=0}^N (1 - \tilde{P}(\frac{1}{n+1}))^{\zeta^{S_{qn}}}$, $B(N) = \prod_{n=1}^N (1 - \tilde{P}(\frac{1}{n}))^{-\zeta^{S_{qn}-1}}$, $C(R, N) = \prod_{r=1}^R \prod_{n=1}^N (1 - \tilde{P}(\frac{1}{nq^r}))^{\zeta^{S_{qn}-r}}$ and $D(R, N) = A(N)B(N)C(R, N)^2$ for given N and R . It shows that, $A(N)$ converges to 0.73, $B(N)$ converges to 1.80, and for all $R > 50$, $C(R, N)$ converges to 0.87, as a consequence, $D(R, N)$ converges to 1. This example can be considered as an evidence to show that the formula (15) holds.

More over, if we suppose, for any j

$$x_j(m) = \begin{cases} \frac{r-1}{r} & \text{if } s_q(m) = j \pmod{r} \\ -\frac{1}{r} & \text{if } s_q(m) \neq j \pmod{r} \end{cases} \quad (16)$$

We clearly have

$$\sum_{j \pmod{r}} x_j(m) = 0 \quad (17)$$

Furthermore,

$$\sum_{j \pmod{r}} x_j(m) \zeta^j = \zeta^{S_q(m)} \quad (18)$$

The Formula (14) can be reformulated as

$$\begin{aligned} \sum_{j \pmod{r}} \zeta^j \sum_{n=0}^{\infty} x_j(n) (\log(1 - \tilde{P}(\frac{1}{n+1}))) + \sum_{r=1}^{\infty} \zeta^{-r} \log(1 - \tilde{P}(\frac{1}{nq^r})) &= \sum_{j \pmod{r}} \zeta^{j-1} \sum_{n=1}^{\infty} x_j(n) (\log(1 - \tilde{P}(\frac{1}{n}))) \\ &+ \sum_{r=1}^{\infty} \zeta^{-r} \log(1 - \tilde{P}(\frac{1}{nq^r})) \end{aligned} \quad (19)$$

Let now η be a primitive root of unity, we can apply relation (19) successively to $\zeta = \eta^a$ for $a = 1, 2, \dots, r-1$.

Because of (17), we also have

$$\begin{aligned} \sum_{j \pmod{r}} \sum_{n=0}^{\infty} x_j(n) (\log(1 - \tilde{P}(\frac{1}{n+1}))) + \sum_{r=1}^{\infty} \zeta^{-r} \log(1 - \tilde{P}(\frac{1}{nq^r})) &= \\ \sum_{j \pmod{r}} \sum_{n=1}^{\infty} x_j(n) (\log(1 - \tilde{P}(\frac{1}{n}))) + \sum_{r=1}^{\infty} \zeta^{-r} \log(1 - \tilde{P}(\frac{1}{nq^r})) &= 0 \end{aligned} \quad (20)$$

Define the matrices : Mat_1 to be $Mat_1 = (\eta^{ij})$ and Mat_2 to be $Mat_2 = (\eta^{ij-i})$, $i = 0, 1, \dots, r-1$; $j = 0, 1, \dots, r-1$, define λ and β by

$$\begin{aligned} \lambda(j) &= \sum_{n=0}^{\infty} x_j(n) (\log(1 - \tilde{P}(\frac{1}{n+1}))) + \sum_{r=1}^{\infty} \zeta^{-r} \log(1 - \tilde{P}(\frac{1}{nq^r})) \\ \beta(j) &= \sum_{n=1}^{\infty} x_j(n) (\log(1 - \tilde{P}(\frac{1}{n}))) + \sum_{r=1}^{\infty} \zeta^{-r} \log(1 - \tilde{P}(\frac{1}{nq^r})) \end{aligned}$$

Let

$$A = \begin{pmatrix} \lambda(0) \\ \lambda(1) \\ \dots \\ \lambda(r-1) \end{pmatrix}$$

$$B = \begin{pmatrix} \beta(0) \\ \beta(1) \\ \dots \\ \beta(r-1) \end{pmatrix}$$

Then we have

$$Mat_1 A = Mat_2 B$$

On the other hand, A is invertible and $Mat_1 = Mat_2 \times Mat_3$ with

$$Mat_3 = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

So we have

$$A = Mat_3 \times B \tag{21}$$

Proposition 5 *We have the equality $\lambda(i) = \beta(i-1)$ for $i = 1, 2, \dots, r-1$ and $\lambda(0) = \beta(r-1)$, which lead to, for $i = 1, 2, \dots, r-1$,*

$$\prod_{n=0}^{\infty} ((1 - \tilde{P}(\frac{1}{n+1}) \times \prod_{r=1}^{\infty} (1 - \tilde{P}(\frac{1}{nq^r})))^{x_j(n)} = \prod_{n=1}^{\infty} ((1 - \tilde{P}(\frac{1}{n}) \times \prod_{r=1}^{\infty} (1 - \tilde{P}(\frac{1}{nq^r})))^{x_{j-1}(n)}$$

and for $i = 0$

$$\prod_{n=0}^{\infty} ((1 - \tilde{P}(\frac{1}{n+1}) \times \prod_{r=1}^{\infty} (1 - \tilde{P}(\frac{1}{nq^r})))^{x_0(n)} = \prod_{n=1}^{\infty} ((1 - \tilde{P}(\frac{1}{n}) \times \prod_{r=1}^{\infty} (1 - \tilde{P}(\frac{1}{nq^r})))^{x_{r-1}(n)}$$

5 Reference

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